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Series concatenation of 2D convolutional codes by means of input-state-output representations

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In this paper we investigate the properties of two-dimensional (2D) convolutional codes which are obtained from series concatenation of two 2D convolutional codes. For this purpose we confine ourselves to dealing with finite-support 2D convolutional codes and make use of the so-called Fornasini-Marchesini input-state-output (ISO) model representations. Within these ISO representations we study when the structural properties of modal reachability and modal observability of the two given ISO representations carry over to the resulting 2D convolutional code. Moreover, we provide necessary conditions for obtaining a systematic concatenated convolutional code. Finally, we present a lower bound on its free distance.

1. Introduction

Codes derived by combining two codes (an inner code and an outer code) form an important class of error-correcting codes called concatenated codes. This class, originally introduced by D. Forney in 1965 (Forney, 1967), became widely used in communications due to fact that this technique results in improving the probability of error (decreasing exponentially with code length), while decoding complexity increases only polynomially (MacWilliams & Sloanepp, 1977, pp. 307–316). Although the first construction of concatenated codes used block codes, NASA started to use a short-constraint-length (64-state) convolutional code as an inner code, decoded by the optimal Viterbi algorithm. Indeed, it was in 1993 that the field of coding theory was revolutionized by the invention of turbo codes (concatenation of two convolutional codes) by Berrou, Glavieux & Thitimajshima (1993). In this paper we are interested in Series Concatenation of Convolutional Codes (SCCC) which are based on the application of two convolutional coding techniques twice on the data input, first on the direct data sequence and second on the interleaved one (Benedetto, Divsalar, Montorsi & Pollara, 1996).

Convolutional codes are *one dimensional* (1D) convolutional codes and can be seen as a generalization of block codes in the sense that a block code is a convolutional code with no delay, i.e., block codes are basically 0D convolutional codes. In this way, *two-dimensional* (2D) convolutional codes extend the 1D convolutional codes. These codes have a practical potential in applications as they are very suitable to encode data recorded in two dimensions, e.g., pictures, storage media, wireless

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applications, etc. Despite the recent increasing interest (Almeida, Napp & Pinto, 2016; Climent, Napp, Pinto & Perea, 2016; Lobo, Bitzer & Vouk, 2012; Napp, Pinto & Simões, 2016; Ozkaya, 2014), in comparison to 1D convolutional codes, little research has been done in the area of 2D convolutional codes and much more needs to be done to make it attractive for real life applications.

Convolutional codes have been defined using different points of view. In this paper we will make use of two: the module-theoretic and the systems theory points of view. The module-theoretic point of view uses generator matrices to represent the convolutional code whereas the systems theory approach uses typically input-state-output representations (Kailath, 1980). Concatenated convolutional codes have traditionally been investigated by means of generator matrices. However, in (Climent, Herranz & Perea 2007, 2008) the first analysis of concatenated convolutional codes using linear systems theory was proposed. The 2D counterpart has been very little investigated (Climent, Napp, Perea & Pinto 2012; Climent et al., 2016; Climent, Napp, Pinto & Simões, 2015; Napp et al. 2016).

In this paper we investigate the properties of the series concatenation of 2D convolutional codes by means of input-state-output representations. We extend previous results presented in (Climent et al., 2015) by studying a new type of concatenation that has not been analysed before in the context of 2D convolutional codes. In this work we confine ourselves to finite-support 2D convolutional codes and make use of the so-called Fornasini-Marchesini input-state-output (ISO) model representations. First we show that the series concatenation of two 2D convolutional codes results in another 2D convolutional code and we explicitly compute an ISO representation. Then, we investigate under which conditions fundamental properties such as modally observability and modally/locally reachability of ISO representations of two 2D convolutional codes carry over after serial concatenation. In fact, we show that while the interconnection of two modally observable 2D systems is also modally observable, the same does not happen for the properties of reachability.

2. Preliminaries

Let $\mathbb{F}$ be a finite field and let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$. Denote by $\mathbb{F}[z_1, z_2]$ the ring of polynomials in two indeterminates with coefficients in $\mathbb{F}$, by $\mathbb{F}(z_1, z_2)$ the field of fractions of $\mathbb{F}[z_1, z_2]$ and by $\mathbb{F}[[z_1, z_2]]$ the ring of formal powers series in two indeterminates with coefficients in $\mathbb{F}$.

2.1 *Polynomial matrices in* $\mathbb{F}[z_1, z_2]$

In this section we start by giving some preliminaries on matrices over the polynomial ring $\mathbb{F}[z_1, z_2]$.

Definition 2.1 (Valcher & Fornasini, 1994)**:** A matrix $M(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$, with $n \geq k$ is,

(a) *unimodular* (i.e., admits a polynomial inverse) if $n = k$ and $\det(M(z_1, z_2)) \in \mathbb{F}\backslash\{0\}$;

(b) *right factor prime* (rFP) if for every factorization

$$M(z_1, z_2) = \overline{M}(z_1, z_2)N(z_1, z_2),$$

with $\overline{M}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$ and $N(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{k \times k}$, $N(z_1, z_2)$ is unimodular;

(c) *right zero prime* (rZP) if the ideal generated by the $k \times k$ minors of $M(z_1, z_2)$ is $\mathbb{F}[z_1, z_2]$.

A matrix is *left factor prime* (ℓFP) / *left zero prime* (ℓZP) if its transpose is rFP / rZP, respectively. When we consider polynomial matrices in one indeterminate, the notions (b) and (c) of the above definition are equivalent. However this is not the case for polynomial matrices in two indeterminates. In fact, zero primeness implies factor primeness, but the contrary does not happen (see Fornasini & Valcher, 1994). The following lemmas give characterizations of right factor primeness and right zero primeness that will be needed later.

Lemma 2.2 (Levy, 1981; Rocha, 1990)**:** *Let $M(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$, with $n \geq k$. Then the following are equivalent:*

(a) *$M(z_1, z_2)$ is rFP;*
(b) *for all $\hat{u}(z_1, z_2) \in \mathbb{F}(z_1, z_2)^k$, $M(z_1, z_2)\hat{u}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^n$ implies that $\hat{u}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^k$.*
(c) *the $k \times k$ minors of $M(z_1, z_2)$ have no non-trivial common factor.*

Lemma 2.3 (Levy, 1981; Rocha, 1990)**:** *Let $M(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$, with $n \geq k$. Then the following are equivalent:*

(a) *$M(z_1, z_2)$ is rZP;*
(b) *$M(z_1, z_2)$ admits a polynomial left inverse;*
(c) *$M(\lambda_1, \lambda_2)$ is full column rank, for all $\lambda_1, \lambda_2 \in \overline{\mathbb{F}}$.*

Remark 1: Obviously, unimodular matrices admit left and right inverses and so by Lemma 2.3 are also rZP and ℓZP and therefore also rFP and ℓFP.

The following lemma will be needed in the sequel. Let $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$, $H(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{(n-k) \times n}$, $n > k$, c_i the ith column of $H(z_1, z_2)$ and r_j the jth row of $G(z_1, z_2)$. We say that the full size minor of $H(z_1, z_2)$ constituted by the columns $c_{i_1}, \ldots, c_{i_{n-k}}$ and the full size minor of $G(z_1, z_2)$ constituted by the rows $r_{j_1}, \ldots, r_{j_k}$ are *corresponding maximal order minors* of $H(z_1, z_2)$ and $G(z_1, z_2)$, if $\{i_1, ..., i_{n-k}\} \cup \{j_1, ..., j_k\} = \{1, \ldots, n\}$ and $\{i_1, ..., i_{n-k}\} \cap \{j_1, ..., j_k\} = \emptyset$.

Lemma 2.4 (Fornasini & Valcher, 1994)**:** *Let $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$ and $H(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{(n-k) \times n}$ be a rFP and a ℓFP matrices, respectively, such that $H(z_1, z_2)G(z_1, z_2) = 0$. Then the corresponding maximal order minors of $H(z_1, z_2)$ and $G(z_1, z_2)$ are equal, modulo a unit of the ring $\mathbb{F}[z_1, z_2]$.*

2.2 2*D linear systems*

Next we give preliminaries on 2D linear systems, which we will use to construct 2D finite support convolutional codes. In particular we consider the Fornasini-Marchesini state space model representation of 2D linear systems (see Fornasini & Marchesini, 1986). In this model a first quarter plane 2D *linear system*, denoted by $\Sigma = (A_1, A_2, B_1, B_2, C, D)$, is given by the updating equations

$$\begin{aligned} x(i+1, j+1) &= A_1 x(i, j+1) + A_2 x(i+1, j) + B_1 u(i, j+1) + B_2 u(i+1, j), \\ y(i, j) &= Cx(i, j) + Du(i, j), \end{aligned} \tag{1}$$

where $A_1, A_2 \in \mathbb{F}^{\delta \times \delta}$, $B_1, B_2 \in \mathbb{F}^{\delta \times k}$, $C \in \mathbb{F}^{(n-k) \times \delta}$, $D \in \mathbb{F}^{(n-k) \times k}$, for $\delta, n, k \in \mathbb{N}$, $n > k$, and with past finite support of the input and of the state and zero initial conditions (i.e., $u(i, j) = 0$, $x(i, j) = 0$ for $i < 0$ or $j < 0$ and $x(0, 0) = 0$). We say that $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ has *dimension* δ. The vectors $x(i, j)$, $u(i, j)$ and $y(i, j)$ represent the *local state, input* and *output* at (i, j), respectively.

The input, state and output 2D sequences (trajectories), $\{u(i, j)\}_{(i,j) \in \mathbb{N}^2}$, $\{x(i, j)\}_{(i,j) \in \mathbb{N}^2}$, $\{y(i, j)\}_{(i,j) \in \mathbb{N}^2}$, respectively, can be represented as formal power series,

$$\hat{u}(z_1, z_2) = \sum_{(i,j) \in \mathbb{N}^2} u(i, j) z_1^i z_2^j \in \mathbb{F}[[z_1, z_2]]^k,$$

$$\hat{x}(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} x(i,j) z_1^i z_2^j \in \mathbb{F}[[z_1, z_2]]^{\delta},$$

$$\hat{y}(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} y(i,j) z_1^i z_2^j \in \mathbb{F}[[z_1, z_2]]^{n-k}.$$

In the sequel we shall use the sequence and the corresponding series interchangeably. Given an input trajectory $\hat{u}(z_1, z_2)$ with corresponding state $\hat{x}(z_1, z_2)$ and output $\hat{y}(z_1, z_2)$ trajectories obtained from (1), the matrix

$$\hat{r}(z_1, z_2) = \begin{bmatrix} \hat{x}(z_1, z_2) \\ \hat{u}(z_1, z_2) \\ \hat{y}(z_1, z_2) \end{bmatrix}$$

is called an *input-state-output trajectory* of Σ. The set of input-state-output trajectories of Σ is given by

$$\ker_{\mathbb{F}[[z_1,z_2]]} X(z_1, z_2) = \left\{ \hat{r}(z_1, z_2) \in \mathbb{F}[[z_1, z_2]]^{n+\delta} \mid X(z_1, z_2)\hat{r}(z_1, z_2) = 0 \right\} \tag{2}$$

where

$$X(z_1, z_2) = \begin{bmatrix} I_\delta - A_1 z_1 - A_2 z_2 & -B_1 z_1 - B_2 z_2 & 0 \\ -C & -D & I_{n-k} \end{bmatrix} \in \mathbb{F}^{(\delta+n-k)\times(\delta+n)}. \tag{3}$$

Next we present reachability and observability properties of such systems.

Definition 2.5 (Fornasini & Marchesini, 1986)**:** Let $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ be a 2D linear system with dimension δ.

(a) Σ is *modally reachable* if the matrix

$$\begin{bmatrix} I_\delta - A_1 z_1 - A_2 z_2 & B_1 z_1 + B_2 z_2 \end{bmatrix}$$

is ℓFP.

(b) Σ is *modally observable* if the matrix

$$\begin{bmatrix} I_\delta - A_1 z_1 - A_2 z_2 \\ C \end{bmatrix}$$

is rFP.

2.3 *2D finite support convolutional codes: ISO representations*

It is well known that a convolutional code is essentially a linear system defined over a finite field. In the 1D case a large body of literature has been devoted to study convolutional codes from a systems theory point of view. In particular special attention has been given to the analysis of convolutional codes by means of input-state-output representations (Rosenthal & York, 1999). Next, we extend this idea to the context of 2D convolutional codes. In this section we recall the definition and properties of 2D finite support convolutional codes and introduce the input-state-output (ISO) representations of such codes by means of the so-called Fornasini-Marchesini state space models.

Definition 2.6 (Valcher & Fornasini, 1994)**:** A 2*D (finite support) convolutional code* $\mathcal{C}$ of rate k/n is a free $\mathbb{F}[z_1, z_2]$-submodule of $\mathbb{F}[z_1, z_2]^n$, where k is the rank of $\mathcal{C}$. A full column rank matrix $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$ whose columns constitute a basis for $\mathcal{C}$, i.e., such that

$$\begin{aligned}\mathcal{C} &= \mathrm{Im}_{\mathbb{F}[z_1,z_2]}\, G(z_1, z_2)\\ &= \left\{\hat{v}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^n \mid \hat{v}(z_1, z_2) = G(z_1, z_2)\hat{u}(z_1, z_2),\ \text{with } \hat{u}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^k\right\},\end{aligned}$$

is called an *encoder* of $\mathcal{C}$. The elements of $\mathcal{C}$ are called *codewords.*

Two full column rank matrices $G(z_1, z_2), \overline{G}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$ are *equivalent encoders* if they generate the same 2D convolutional code, i.e., if

$$\mathrm{Im}_{\mathbb{F}[z_1,z_2]}\, G(z_1, z_2) = \mathrm{Im}_{\mathbb{F}[z_1,z_2]}\, \overline{G}(z_1, z_2),$$

which happens if and only if there exists a unimodular matrix $U(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{k\times k}$ such that $G(z_1, z_2)U(z_1, z_2) = \overline{G}(z_1, z_2)$ (see Valcher & Fornasini, 1994).

Note that the fact that two equivalent encoders differ by unimodular matrices also implies that the primeness properties of the encoders of a code are preserved, i.e., if $\mathcal{C}$ admits a rFP (rZP) encoder then all its encoders are rFP (rZP). A 2D finite support convolutional code $\mathcal{C}$ that admits rFP encoders is called *noncatastrophic*, and it is named *basic* if all its encoders are rZP. An encoder of the form

$$G(z_1, z_2) = \begin{bmatrix}\tilde{G}(z_1, z_2)\\ I_k\end{bmatrix} \in \mathbb{F}[z_1, z_2]^{n\times k},$$

up to a row permutation is called *systematic*. Not all 2D convolutional codes admit a systematic encoder. We call 2D systematic code to a 2D convolutional code that admits a systematic encoder. The class of 2D systematic codes is contained in the class of the 2D basic convolutional codes as the following lemma shows. The proof is straightforward and we omit it.

Lemma 2.7: *Let $\mathcal{C}$ be a 2D convolutional code with encoder $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$. Then $\mathcal{C}$ is systematic if and only if $G(z_1, z_2)$ admits a nonzero constant full size minor.*

An important measure of robustness of a code is its distance. We define the notion of distance as in Weiner (1998). The weight of

$$\hat{v}(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} v(i, j) z_1^i z_2^j \in \mathbb{F}[z_1, z_2]^n,$$

with $v(i, j) \in \mathbb{F}^n$ for $(i, j) \in \mathbb{N}^2$, is given by

$$\mathrm{wt}(\hat{v}) = \sum_{(i,j)\in\mathbb{N}^2} \mathrm{wt}(v(i, j)),$$

where $\mathrm{wt}(v(i, j))$ is the number of nonzero elements of $v(i, j)$. The distance between $\hat{v}_1(z_1, z_2), \hat{v}_2(z_1, z_2) \in \mathbb{F}[z_1, z_2]^n$ is $\mathrm{dist}(\hat{v}_1, \hat{v}_2) = \mathrm{wt}(\hat{v}_1 - \hat{v}_2)$.

Definition 2.8: Given a 2D convolutional code $\mathcal{C}$, the *distance* of $\mathcal{C}$, denoted by $\mathrm{dist}(\mathcal{C})$ is defined as

$$\min \left\{ \mathrm{dist}(\hat{v}_1, \hat{v}_2) \mid \hat{v}_1(z_1, z_2), \hat{v}_2(z_1, z_2) \in \mathcal{C}, \text{ with } \hat{v}_1(z_1, z_2) \neq \hat{v}_2(z_1, z_2) \right\}.$$

Note that the linearity of $\mathcal{C}$ implies that $\mathrm{dist}(\mathcal{C}) = \min \{\mathrm{wt}(\hat{v}) \mid \hat{v}(z_1, z_2) \in \mathcal{C}, \text{ with } \hat{v}(z_1, z_2) \neq 0\}$.

Next we make use of the representation machinery in 2D linear systems to treat 2D convolutional codes. We consider a first quarter plane 2D linear system Σ as defined in (1). For $(i, j) \in \mathbb{N}^2$, define

$$v(i, j) = \begin{bmatrix} y(i, j) \\ u(i, j) \end{bmatrix} \in \mathbb{F}^n$$

to be the *code vector.*

We will only consider the finite support input-output trajectories, $\{v(i, j)\}_{(i,j)\in\mathbb{N}^2}$ of (1). Moreover, we will not consider such vectors with the corresponding state vector $\hat{x}(z_1, z_2)$ having infinite support, since this would make the system remain indefinitely excited. Thus, we will restrict ourselves to finite support input-output trajectories $(\hat{u}(z_1, z_2), \hat{y}(z_1, z_2))$ with corresponding state $\hat{x}(z_1, z_2)$ also having finite support. We call such trajectories $(\hat{u}(z_1, z_2), \hat{y}(z_1, z_2))$ *finite-weight input-output trajectories* and the triple $(\hat{x}(z_1, z_2), \hat{u}(z_1, z_2), \hat{y}(z_1, z_2))$ *finite-weight trajectories.* Note that not all finite support input-output trajectories have corresponding state $\hat{x}(z_1, z_2)$ also having finite support. The following result asserts that the set of finite-weight trajectories of (1) forms a 2D finite support convolutional code.

Theorem 2.9 (Napp et al., 2010)**:** *The set of finite-weight input-output trajectories of* (1) *is a* 2D *finite support convolutional code of rate* k/n.

It is worth mentioning that this approach is different from the one adopted in Fornasini and Valcher (1994) where the codewords are constituted only by the output $\hat{y}(z_1, z_2)$ of a system.

We denote by $\mathcal{C}(A_1, A_2, B_1, B_2, C, D)$ the 2D finite support convolutional code whose codewords are the finite-weight input-output trajectories of the 2D linear system $\Sigma = (A_1, A_2, B_1, B_2, C, D)$. Moreover, Σ is called an *input-state-output (ISO) representation* of $\mathcal{C}(A_1, A_2, B_1, B_2, C, D)$ (see Napp et al., 2010). The input vector, the output vector and the code vector associated to a finite-weight trajectory of Σ are called *information vector*, *parity vector* and *codeword* of $\mathcal{C}$, respectively.

Next we will show how the properties of reachability and observability of ISO representations, stated in Definition 2.5, reflect on the structure of the corresponding code.

Theorem 2.10 (Napp et al., 2010)**:** *Let* $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ *be a* 2D *linear system. If* Σ *is modally observable then* $\mathcal{C}(A_1, A_2, B_1, B_2, C, D)$ *is noncatastrophic and its codewords are the finite support input-output trajectories of* Σ.

In case the ISO representation is modally reachable a necessary and sufficient condition can be stated for the noncatastrophicity of the corresponding code. To show that we need first to introduce the following technical lemma.

Lemma 2.11 (Climent et al., 2015)**:** *Let* $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ *be a* 2D *linear system and* $X(z_1, z_2)$ *the corresponding matrix defined in* (3). *Then* Σ *is modally reachable if and only if the matrix* $X(z_1, z_2)$ *is* ℓFP.

Proof. Suppose that Σ is modally reachable; then $\begin{bmatrix} I_\delta - A_1 z_1 - A_2 z_2 & B_1 z_1 + B_2 z_2 & 0 \end{bmatrix}$ is ℓFP.

Let $\hat{w}_1(z_1,z_2) \in \mathbb{F}(z_1,z_2)^{\delta}$ and $\hat{w}_2(z_1,z_2) \in \mathbb{F}(z_1,z_2)^{n-k}$ be such that

$$\begin{bmatrix}\hat{w}_1(z_1,z_2)^T & \hat{w}_2(z_1,z_2)^T\end{bmatrix} X(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{1\times(\delta+n)}.$$

Then $\begin{bmatrix}\hat{w}_1(z_1,z_2)^T & \hat{w}_2(z_1,z_2)^T\end{bmatrix}\begin{bmatrix}0\\ I_{n-k}\end{bmatrix} = \hat{w}_2(z_1,z_2)^T \in \mathbb{F}[z_1,z_2]^{1\times(n-k)}$. Consequently,

$$\hat{w}_1(z_1,z_2)^T\begin{bmatrix}I_\delta - A_1z_1 - A_2z_2 & -B_1z_1 - B_2z_2 & 0\end{bmatrix} \in \mathbb{F}[z_1,z_2]^{1\times(\delta+n)}.$$

Therefore, by Lemma 2.2, $\hat{w}_1(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{\delta}$ and $X(z_1,z_2)$ is ℓFP.

Now suppose that $X(z_1,z_2)$ is ℓFP. Let $\hat{w}_1(z_1,z_2) \in \mathbb{F}(z_1,z_2)^{\delta}$ be such that

$$\hat{w}_1(z_1,z_2)^T\begin{bmatrix}I_\delta - A_1z_1 - A_2z_2 & B_1z_1 + B_2z_2\end{bmatrix} \in \mathbb{F}[z_1,z_2]^{1\times(\delta+k)}$$

then

$$\hat{w}_1(z_1,z_2)^T\begin{bmatrix}I_\delta - A_1z_1 - A_2z_2 & -B_1z_1 - B_2z_2 & 0\end{bmatrix} \in \mathbb{F}[z_1,z_2]^{1\times(\delta+n)}.$$

Let us consider $\hat{w}_2(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{n-k}$; then

$$\hat{w}_2(z_1,z_2)^T\begin{bmatrix}-C & -D & I_{n-k}\end{bmatrix} \in \mathbb{F}[z_1,z_2]^{1\times(\delta+n)}$$

and therefore

$$\begin{bmatrix}\hat{w}_1(z_1,z_2)^T & \hat{w}_2(z_1,z_2)^T\end{bmatrix} X(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{1\times(\delta+n)},$$

and, since $X(z_1,z_2)$ is ℓFP, by Lemma 2.2, $\begin{bmatrix}\hat{w}_1(z_1,z_2)^T & \hat{w}_2(z_1,z_2)^T\end{bmatrix} \in \mathbb{F}[z_1,z_2]^{1\times(\delta+n-k)}$ and therefore $\hat{w}_1(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{\delta}$. The result follows from Lemma 2.2. □

Theorem 2.12 (Climent et al., 2015)**:** *Let $\Sigma = (A_1,A_2,B_1,B_2,C,D)$ be a modally reachable 2D linear system. Then Σ is modally observable if and only if $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$ is noncatastrophic.*

Proof. From Theorem 2.10, we just need to prove that if $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$ is noncatastrophic then Σ is modally observable. Let us assume that Σ is not modally observable. Then, from Lemma 2.2, there exists a nonconstant $d(z_1,z_2) \in \mathbb{F}[z_1,z_2]$ which is a common factor of all $\delta\times\delta$ minors of

$$\begin{bmatrix}I_\delta - A_1z_1 - A_2z_2\\ -C\end{bmatrix}.$$

Let $L(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{\delta\times k}$ and $G(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{n\times k}$ be such that

$$X(z_1,z_2)\begin{bmatrix}L(z_1,z_2)\\ G(z_1,z_2)\end{bmatrix} = 0,$$

with $X(z_1,z_2)$ defined in (3) and where $\begin{bmatrix}L(z_1,z_2)\\ G(z_1,z_2)\end{bmatrix}$ is rFP and $G(z_1,z_2)$ is an encoder of $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$ (see Napp et al., 2010, proof of Theorem 1).

From Lemma 2.11, $X(z_1,z_2)$ is ℓFP and note that all $(\delta+n-k)\times(\delta+n-k)$ minors of $X(z_1,z_2)$ whose corresponding submatrices include $\begin{bmatrix} I_\delta - A_1z_1 - A_2z_2 \\ -C \end{bmatrix}$ have also $d(z_1,z_2)$ as common factor. Therefore, by Lemma 2.4, all $k\times k$ minors of $G(z_1,z_2)$ have $d(z_1,z_2)$ as common factor which implies that $G(z_1,z_2)$ is not rFP and consequently $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$ is catastrophic. □

The next proposition establishes necessary and sufficient conditions for a convolutional codes to be systematic.

Proposition 2.13: *Let $\Sigma = (A_1,A_2,B_1,B_2,C,D)$ be a modally reachable 2D linear system and $X(z_1,z_2)$ the corresponding matrix defined in (3). Then $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$ is systematic if and only if $X(z_1,z_2)$ has a $(\delta+n-k)\times(\delta+n-k)$ unimodular submatrix, computed by picking up necessarily its first δ columns.*

Proof. Let $L(z_1,z_2) = \begin{bmatrix} L_1(z_1,z_2) \\ G(z_1,z_2) \end{bmatrix}$ be a rFP matrix such that $X(z_1,z_2)L(z_1,z_2) = 0$, with $L_1(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{\delta\times k}$ and $G(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{n\times k}$ an encoder of $\mathcal{C} = \mathcal{C}(A_1,A_2,B_1,B_2,C,D)$. Note that, since Σ is modally reachable then, by Lemma 2.11, the matrix $X(z_1,z_2)$ is ℓFP.

Then $\mathcal{C}$ is systematic if and only if, by Lemma 2.7, $G(z_1,z_2)$ admits a nonzero constant full size minor, i.e., if and only if, by Lemma 2.4, $X(z_1,z_2)$ has a nonzero constant $(\delta+n-k)\times(\delta+n-k)$ minor, computed by picking up necessarily its first δ columns, and the result follows. □

For a given system $\Sigma = (A_1,A_2,B_1,B_2,C,D)$ of dimension δ the property of $I_\delta - A_1z_1 - A_2z_2$ being unimodular guarantees that such a system is modally reachable and modally observable and the corresponding convolutional code is systematic. The proof is simple but we include it for the sake of completeness.

Proposition 2.14: *Let $\Sigma = (A_1,A_2,B_1,B_2,C,D)$ be a 2D linear system such that $I_\delta - A_1z_1 - A_2z_2$ is unimodular. Then:*

(1) Σ is modally reachable and modally observable.
(2) $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$ is systematic, with an encoder

$$G(z_1,z_2) = \begin{bmatrix} T(z_1,z_2) \\ I_k \end{bmatrix}$$

where $T(z_1,z_2) = C(I_\delta - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2) + D \in \mathbb{F}[z_1,z_2]^{(n-k)\times k}$.

Proof. (1) If $I_\delta - A_1z_1 - A_2z_2$ is unimodular, then $I_\delta - A_1z_1 - A_2z_2$ is ℓZP and therefore $\begin{bmatrix} I_\delta - A_1z_1 - A_2z_2 & B_1z_1 + B_2z_2 \end{bmatrix}$ is ℓZP which means that it is ℓFP. Thus Σ is modally reachable. By Lemma 2.11, the corresponding matrix $X(z_1,z_2)$ defined in (3) is ℓFP and all his $\delta\times\delta$ minors have no common factors.

On the other hand, let $L(z_1,z_2) = \begin{bmatrix} L_1(z_1,z_2) \\ G(z_1,z_2) \end{bmatrix}$ be a rFP matrix such that $X(z_1,z_2)L(z_1,z_2) = 0$, with $L_1(z_1,z_2) \in \mathbb{F}^{\delta\times k}$ and $G(z_1,z_2) \in \mathbb{F}^{n\times k}$ an encoder of $\mathcal{C} = \mathcal{C}(A_1,A_2,B_1,B_2,C,D)$. Then by Lemma 2.4, we know that the $k\times k$ minors of $G(z_1,z_2)$ have no common factors and the result follows from Theorem 2.12.

(2) The result follows from (1) and Proposition 2.13. If we re-write the code vector as a formal power series it is easy to see that

$$\hat{v}(z_1,z_2) = \begin{bmatrix} T(z_1,z_2) \\ I_k \end{bmatrix} \hat{u}(z_1,z_2),$$

where $T(z_1, z_2) = C(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2) + D \in \mathbb{F}(z_1, z_2)^{(n-k)\times k}$. In fact, since an input-output trajectory of Σ satisfies

$$\begin{bmatrix} I_\delta - A_1z_1 - A_2z_2 & -B_1z_1 - B_2z_2 & 0 \\ -C & -D & I_{n-k} \end{bmatrix} \begin{bmatrix} \hat{x}(z_1, z_2) \\ \hat{u}(z_1, z_2) \\ \hat{y}(z_1, z_2) \end{bmatrix} = 0,$$

then

$$\begin{cases} (I_\delta - A_1z_1 - A_2z_2)\hat{x}(z_1, z_2) - (B_1z_1 + B_2z_2)\hat{u}(z_1, z_2) = 0 \\ -C\hat{x}(z_1, z_2) - D\hat{u}(z_1, z_2) + I_{n-k}\hat{y}(z_1, z_2) = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} \hat{x}(z_1, z_2) = (I_\delta - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2)\hat{u}(z_1, z_2) \\ \hat{y}(z_1, z_2) = \left(C(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2) + D\right)\hat{u}(z_1, z_2) \end{cases}.$$

Therefore

$$\hat{v}(z_1, z_2) = \begin{bmatrix} \hat{y}(z_1, z_2) \\ \hat{u}(z_1, z_2) \end{bmatrix} = \begin{bmatrix} T(z_1, z_2) \\ I_k \end{bmatrix} \hat{u}(z_1, z_2).$$

□

3. ISO representations of concatenated 2D convolutional codes

In this section we study 2D convolutional codes that result from series concatenation of other two 2D convolutional codes. We will consider a very general series concatenation scheme as the one proposed in Climent et al. (2007) for series concatenation f 1D convolutional codes. In particular we focus on finding conditions for the properties of modal reachability and modal observability for obtaining a systematic concatenated code. We conclude the section by giving a lower bound on the distance of the resulting code.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two 2D convolutional codes of rate k/m and m/n, respectively. We denote by $u^{(i)}$, $y^{(i)}$ and $v^{(i)}$ the information vector, parity vector and codeword of $\mathcal{C}_i$, for $i = 1, 2$, respectively.

Let us consider the series concatenation of $\mathcal{C}_1$ and $\mathcal{C}_2$ so that the information vector $u^{(2)}$ of $\mathcal{C}_2$ is the codeword of $\mathcal{C}_1$, i.e.

$$u^{(2)} = v^{(1)} = \begin{bmatrix} y^{(1)} \\ u^{(1)} \end{bmatrix}$$

as represented in Figure 1.

The next result shows that the series concatenation of two 2D convolutional codes is a 2D convolutional code and presents an ISO representation for this concatenation.

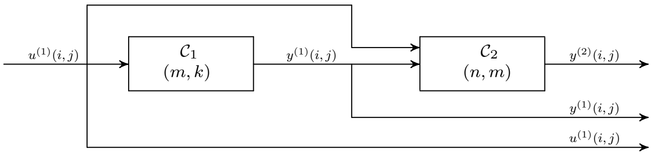

Figure 1. Series concatenation of $\mathcal{C}_1$ and $\mathcal{C}_2$

Theorem 3.1 (Climent et al., 2015)**:** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two 2D convolutional codes of rate k/m and m/n, respectively, and for $i = 1, 2$ let*

$$\Sigma_i = \left(A_1^{(i)}, A_2^{(i)}, B_1^{(i)}, B_2^{(i)}, C^{(i)}, D^{(i)}\right)$$

be an ISO representation of $\mathcal{C}_i$ of dimension δ_i.

The series concatenation $\mathcal{C}$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ is a 2D convolutional code of rate k/n with ISO representation

$$\Sigma = (A_1, A_2, B_1, B_2, C, D),$$

given by

$$A_1 = \begin{bmatrix} A_1^{(2)} & B_{11}^{(2)} C^{(1)} \\ 0 & A_1^{(1)} \end{bmatrix} \in \mathbb{F}^{(\delta_1+\delta_2)\times(\delta_1+\delta_2)}, \quad A_2 = \begin{bmatrix} A_2^{(2)} & B_{21}^{(2)} C^{(1)} \\ 0 & A_2^{(1)} \end{bmatrix} \in \mathbb{F}^{(\delta_1+\delta_2)\times(\delta_1+\delta_2)},$$

$$B_1 = \begin{bmatrix} B_{12}^{(2)} + B_{11}^{(2)} D^{(1)} \\ B_1^{(1)} \end{bmatrix} \in \mathbb{F}^{(\delta_1+\delta_2)\times k}, \quad B_2 = \begin{bmatrix} B_{22}^{(2)} + B_{21}^{(2)} D^{(1)} \\ B_2^{(1)} \end{bmatrix} \in \mathbb{F}^{(\delta_1+\delta_2)\times k},$$

$$C = \begin{bmatrix} C^{(2)} & D_1^{(2)} C^{(1)} \\ 0 & C^{(1)} \end{bmatrix} \in \mathbb{F}^{(n-k)\times(\delta_1+\delta_2)}, \quad D = \begin{bmatrix} D_1^{(2)} D^{(1)} + D_2^{(2)} \\ D^{(1)} \end{bmatrix} \in \mathbb{F}^{(n-k)\times k},$$

where $B_1^{(2)} = \begin{bmatrix} B_{11}^{(2)} & B_{12}^{(2)} \end{bmatrix}$, $B_2^{(2)} = \begin{bmatrix} B_{21}^{(2)} & B_{22}^{(2)} \end{bmatrix}$ and $D^{(2)} = \begin{bmatrix} D_1^{(2)} & D_2^{(2)} \end{bmatrix}$, with $B_{11}^{(2)} \in \mathbb{F}^{\delta_2\times(m-k)}$, $B_{12}^{(2)} \in \mathbb{F}^{\delta_2\times k}$, $D_1^{(2)} \in \mathbb{F}^{(n-m)\times(m-k)}$ and $D_2^{(2)} \in \mathbb{F}^{(n-m)\times k}$.

Proof. Let us consider Σ_1 and Σ_2 the ISO representations of $\mathcal{C}_1$ and $\mathcal{C}_2$ given, respectively, by

$$\begin{aligned} x^{(1)}(i+1, j+1) &= A_1^{(1)} x^{(1)}(i, j+1) + A_2^{(1)} x^{(1)}(i+1, j) + B_1^{(1)} u^{(1)}(i, j+1) + B_2^{(1)} u^{(1)}(i+1, j), \\ y^{(1)}(i,j) &= C^{(1)} x^{(1)}(i,j) + D^{(1)} u^{(1)}(i,j), \end{aligned}$$

and

$$\begin{aligned} x^{(2)}(i+1, j+1) &= A_1^{(2)} x^{(2)}(i, j+1) + A_2^{(2)} x^{(2)}(i+1, j) + B_1^{(2)} u^{(2)}(i, j+1) + B_2^{(2)} u^{(2)}(i+1, j), \\ y^{(2)}(i,j) &= C^{(2)} x^{(2)}(i,j) + D^{(2)} u^{(2)}(i,j), \end{aligned}$$

Bearing in mind that the information vector of $\mathcal{C}_2$ is the codeword of $\mathcal{C}_1$, we can replace in Σ_2 the input vector $u^{(2)}(i,j)$ by

$$v^{(1)}(i,j) = \begin{bmatrix} y^{(1)}(i,j) \\ u^{(1)}(i,j) \end{bmatrix}$$

and we obtain

$$\begin{aligned} x^{(2)}(i+1,j+1) &= \begin{bmatrix} A_1^{(2)} & B_{11}^{(2)}C^{(1)} \end{bmatrix} \begin{bmatrix} x^{(2)}(i,j+1) \\ x^{(1)}(i,j+1) \end{bmatrix} + \begin{bmatrix} A_2^{(2)} & B_{21}^{(2)}C^{(1)} \end{bmatrix} \begin{bmatrix} x^{(2)}(i+1,j) \\ x^{(1)}(i+1,j) \end{bmatrix} + \\ &+ \left(B_{12}^{(2)} + B_{11}^{(2)}D^{(1)}\right) u^{(1)}(i,j+1) + \left(B_{22}^{(2)} + B_{21}^{(2)}D^{(1)}\right) u^{(1)}(i+1,j), \\ y^{(2)}(i,j) &= \begin{bmatrix} D_1^{(2)}C^{(1)} & C^{(2)} \end{bmatrix} \begin{bmatrix} x^{(2)}(i,j) \\ x^{(1)}(i,j) \end{bmatrix} + \left(D_1^{(2)}D^{(1)} + D_2^{(2)}\right) u^{(1)}(i,j), \end{aligned}$$

where $B_1^{(2)} = \begin{bmatrix} B_{11}^{(2)} & B_{12}^{(2)} \end{bmatrix}$, $B_2^{(2)} = \begin{bmatrix} B_{21}^{(2)} & B_{22}^{(2)} \end{bmatrix}$ and $D^{(2)} = \begin{bmatrix} D_1^{(2)} & D_2^{(2)} \end{bmatrix}$, with $B_{11}^{(2)} \in \mathbb{F}^{\delta_2 \times (m-k)}$, $B_{12}^{(2)} \in \mathbb{F}^{\delta_2 \times k}$, $D_1^{(2)} \in \mathbb{F}^{(n-m)\times(m-k)}$ and $D_2^{(2)} \in \mathbb{F}^{(n-m)\times k}$.

Note that the input, state and output vectors of the ISO representation of the series concatenation of $\mathcal{C}_1$ and $\mathcal{C}_2$ are, respectively,

$$u(i,j) = u^{(1)}(i,j), \quad x(i,j) = \begin{bmatrix} x^{(2)}(i,j) \\ x^{(1)}(i,j) \end{bmatrix}, \quad y(i,j) = \begin{bmatrix} y^{(2)}(i,j) \\ y^{(1)}(i,j) \end{bmatrix}.$$

Then the ISO representation of the series concatenation of $\mathcal{C}_1$ and $\mathcal{C}_2$ is

$$\begin{aligned} x(i+1,j+1) &= \begin{bmatrix} A_1^{(2)} & B_{11}^{(2)}C^{(1)} \\ 0 & A_1^{(1)} \end{bmatrix} x(i,j+1) + \begin{bmatrix} A_2^{(2)} & B_{21}^{(2)}C^{(1)} \\ 0 & A_2^{(1)} \end{bmatrix} x(i+1,j) + \\ &+ \begin{bmatrix} B_{12}^{(2)} + B_{11}^{(2)}D^{(1)} \\ B_1^{(1)} \end{bmatrix} u(i,j+1) + \begin{bmatrix} B_{22}^{(2)} + B_{21}^{(2)}D^{(1)} \\ B_2^{(1)} \end{bmatrix} u(i+1,j), \\ y(i,j) &= \begin{bmatrix} C^{(2)} & D_1^{(2)}C^{(1)} \\ 0 & C^{(1)} \end{bmatrix} x(i,j) + \begin{bmatrix} D_1^{(2)}D^{(1)} + D_2^{(2)} \\ D^{(1)} \end{bmatrix} u(i,j). \end{aligned}$$

□

It is natural to ask when a code obtained by this concatenation is modally observable. A sufficient condition is given by the following theorem.

Theorem 3.2 (Climent et al., 2015)**:** *For* $i = 1,2$, *let* $\Sigma_i = \left(A_1^{(i)}, A_2^{(i)}, B_1^{(i)}, B_2^{(i)}, C^{(i)}, D^{(i)}\right)$ *be a* 2*D linear system of dimension* δ_i. *If* Σ_1 *and* Σ_2 *are modally observable, then the* 2*D linear system* Σ *defined in Theorem* 3.1 *is modally observable.*

Proof. Assume that Σ_1 and Σ_2 are modally observable. Attending to Theorem 3.1, we have to prove that the matrix

$$Y(z_1,z_2)=\begin{bmatrix} I_{\delta_2}-A_1^{(2)}z_1-A_2^{(2)}z_2 & -B_{11}^{(2)}C^{(1)}z_1-B_{21}^{(2)}C^{(1)}z_2 \\ 0 & I_{\delta_1}-A_1^{(1)}z_1-A_2^{(1)}z_2 \\ C^{(2)} & D_1^{(2)}C^{(1)} \\ 0 & C^{(1)} \end{bmatrix}$$

is rFP. Let $\hat{w}(z_1,z_2)\in\mathbb{F}(z_1,z_2)^{\delta_1+\delta_2}$ be such that

$$Y(z_1,z_2)\hat{w}(z_1,z_2)\in\mathbb{F}[z_1,z_2]^{\delta_1+\delta_2+n-k}.$$

Suppose that $\hat{w}(z_1,z_2)=\begin{bmatrix}\hat{w}_2(z_1,z_2)^T & \hat{w}_1(z_1,z_2)^T\end{bmatrix}^T$ with $\hat{w}_2(z_1,z_2)\in\mathbb{F}(z_1,z_2)^{\delta_2}$. Then

$$\begin{bmatrix} I_{\delta_1}-A_1^{(1)}z_1-A_2^{(1)}z_2 \\ C^{(1)} \end{bmatrix}\hat{w}_1(z_1,z_2)\in\mathbb{F}[z_1,z_2]^{\delta_1+m-k}$$

and, since Σ_1 is modally observable, $\hat{w}_1(z_1,z_2)\in\mathbb{F}[z_1,z_2]^{\delta_1}$.

On the other hand,

$$\begin{bmatrix} I_{\delta_2}-A_1^{(2)}z_1-A_2^{(2)}z_2 \\ C^{(2)} \end{bmatrix}\hat{w}_2(z_1,z_2) + \begin{bmatrix} -B_{11}^{(2)}C^{(1)}z_1-B_{21}^{(2)}C^{(1)}z_2 \\ D_1^{(2)}C^{(1)} \end{bmatrix}\hat{w}_1(z_1,z_2)\in\mathbb{F}[z_1,z_2]^{\delta_2+n-m}$$

which implies that

$$\begin{bmatrix} I_{\delta_2}-A_1^{(2)}z_1-A_2^{(2)}z_2 \\ C^{(2)} \end{bmatrix}\hat{w}_2(z_1,z_2)\in\mathbb{F}[z_1,z_2]^{\delta_2+n-m}$$

and, since Σ_2 is modally observable, $\hat{w}_2(z_1,z_2)\in\mathbb{F}[z_1,z_2]^{\delta_2}$, and therefore $\hat{w}(z_1,z_2)\in\mathbb{F}[z_1,z_2]^{\delta_1+\delta_2}$. Thus, by Lemma 2.2, $Y(z_1,z_2)$ is rFP and therefore Σ is modally observable. □

The next corollary is a consequence of Theorems 2.12 and 3.2 and shows that if the original systems are modally reachable then modal observability and noncatastrophicity carry over to the resulting concatenated code.

Corollary 3.3: *For $i=1,2$, let $\mathcal{C}_i$ be a 2D convolutional code with ISO representation Σ_i and such that Σ_i are modally reachable. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are noncatastrophic then the 2D linear system $\Sigma=(A_1,A_2,B_1,B_2,C,D)$ defined in Theorem 3.1 is modally observable and therefore the respective series concatenated code $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$ is noncatastrophic.*

The next example shows that it is not sufficient that the 2D linear systems Σ_1 and Σ_2 are modally reachable to get the 2D linear system defined in Theorem 3.1 modally reachable.

Example 3.4: Let α be a primitive element of the Galois field $\mathbb{F} = GF(8)$ with $\alpha^3 + \alpha + 1 = 0$, and consider, for $i = 1, 2$, the 2D linear system $\Sigma_i = \left(A_1^{(i)}, A_2^{(i)}, B_1^{(i)}, B_2^{(i)}, C^{(i)}, D^{(i)}\right)$, where

$$A_1^{(1)} = A_2^{(1)} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{bmatrix}, \qquad B_1^{(1)} = B_2^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha^6 \end{bmatrix},$$

$$C^{(1)} = \begin{bmatrix} \alpha^4 & \alpha^3 \end{bmatrix}, \qquad D^{(1)} = \begin{bmatrix} 1 & \alpha^4 \end{bmatrix},$$

$$A_1^{(2)} = A_2^{(2)} = \begin{bmatrix} \alpha^4 & 1 \\ \alpha^3 & 0 \end{bmatrix}, \qquad B_1^{(2)} = B_2^{(2)} = \begin{bmatrix} 1 & 0 & 1 \\ \alpha & 1 & \alpha \end{bmatrix},$$

and $C^{(2)}$ and $D^{(2)}$ are matrices of suitable dimensions, and let $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ be the 2D linear system as defined in Theorem 3.1, with

$$A_1 = A_2 = \begin{bmatrix} \alpha^4 & 1 & \alpha^4 & \alpha^3 \\ \alpha^3 & 0 & \alpha^5 & \alpha^4 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha^2 \end{bmatrix} \text{ and } B_1 = B_2 = \begin{bmatrix} 1 & \alpha^5 \\ \alpha^3 & \alpha^6 \\ 1 & 0 \\ 0 & \alpha^6 \end{bmatrix}.$$

It is easy to see that the matrices

$$R^{(1)}(z_1, z_2) = \begin{bmatrix} I_2 - A_1^{(1)} z_1 - A_2^{(1)} z_2 & B_1^{(1)} z_1 + B_2^{(1)} z_2 \end{bmatrix}$$

and

$$R^{(2)}(z_1, z_2) = \begin{bmatrix} I_2 - A_1^{(2)} z_1 - A_2^{(2)} z_2 & B_1^{(2)} z_1 + B_2^{(2)} z_2 \end{bmatrix}$$

are ℓFP. In fact,

$$R^{(1)}(z_1, z_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & 0 \\ 0 & \alpha^3 \end{bmatrix} = I_2, \qquad R^{(2)}(z_1, z_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha^4 & 1 \\ \alpha^2 & \alpha \\ 0 & 0 \end{bmatrix} = I_2,$$

which means that $R^{(1)}(z_1, z_2)$ and $R^{(2)}(z_1, z_2)$ are ℓZP, and therefore, they are also ℓFP.

But the matrix

$$R(z_1, z_2) = \begin{bmatrix} I_4 - A_1 z_1 - A_2 z_2 & B_1 z_1 + B_2 z_2 \end{bmatrix}$$

is not ℓFP. In fact, there exists

$$\hat{w}(z_1, z_2) \in \mathbb{F}(z_1, z_2)^{1\times 4} \setminus \mathbb{F}[z_1, z_2]^{1\times 4}$$

such that

$$\hat{w}(z_1,z_2)R(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{1\times 6}.$$

Just consider

$$\hat{w}(z_1,z_2) = \frac{1}{1+\alpha(z_1+z_2)}\begin{bmatrix}1 & z_1+z_2 & \alpha(z_1+z_2)^2 & 0\end{bmatrix},$$

which is not polynomial, and

$$\begin{aligned}\hat{w}(z_1,z_2)R(z_1,z_2) = \big[1+\alpha^2(z_1+z_2) \quad & 0 \quad \alpha^4(z_1+z_2)(1+\alpha^4(z_1+z_2)) \\ & \alpha^3(z_1+z_2) \quad (z_1+z_2)(1+z_1+z_2) \quad \alpha^5(z_1+z_2)\big],\end{aligned}$$

which is polynomial. Then $R(z_1,z_2)$ is not ℓFP, which means that Σ is not modally reachable.

Next we present a necessary condition for the concatenated code to be modally reachable.

Theorem 3.5: *For $i=1,2$, let $\Sigma_i = \left(A_1^{(i)}, A_2^{(i)}, B_1^{(i)}, B_2^{(i)}, C^{(i)}, D^{(i)}\right)$ be a 2D linear system of dimension δ_i, such that the matrix $I_{\delta_2} - A_1^{(2)}z_1 - A_2^{(2)}z_2$ is unimodular. Let Σ be the 2D linear system defined in Theorem 3.1. Then:*

(1) If Σ_1 is modally reachable, then Σ is modally reachable.
(2) If Σ_1 is modally observable, then Σ is modally observable.

Proof. (1) Assume that Σ_1 is modally reachable and that the matrix $I_{\delta_2} - A_1^{(2)}z_1 - A_2^{(2)}z_2$ is unimodular. According to Theorem 3.1, we have to prove that the matrix $R(z_1,z_2)$ given by

$$\begin{bmatrix} I_{\delta_2} - A_1^{(2)}z_1 - A_2^{(2)}z_2 & -B_{11}^{(2)}C^{(1)}z_1 - B_{21}^{(2)}C^{(1)}z_2 & \left(B_{12}^{(2)} + B_{11}^{(2)}D^{(1)}\right)z_1 + \left(B_{22}^{(2)} + B_{21}^{(2)}D^{(1)}\right)z_2 \\ 0 & I_{\delta_1} - A_1^{(1)}z_1 - A_2^{(1)}z_2 & B_1^{(1)}z_1 + B_2^{(1)}z_2 \end{bmatrix}$$

is ℓFP.

Let $\hat{w}(z_1,z_2) \in \mathbb{F}(z_1,z_2)^{1\times(\delta_1+\delta_2)}$ be such that $\hat{w}(z_1,z_2)R(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{1\times(\delta_1+\delta_2+k)}$. Suppose that $\hat{w}(z_1,z_2) = \begin{bmatrix}\hat{w}_2(z_1,z_2)^T & \hat{w}_1(z_1,z_2)^T\end{bmatrix}$ with $\hat{w}_2(z_1,z_2) \in \mathbb{F}(z_1,z_2)^{\delta_2}$. Then

$$\hat{w}_2(z_1,z_2)^T\left(I_{\delta_2} - A_1^{(2)}z_1 - A_2^{(2)}z_2\right) \in \mathbb{F}[z_1,z_2]^{1\times\delta_2}$$

and, since $I_{\delta_2} - A_1^{(2)}z_1 - A_2^{(2)}z_2$ is unimodular by Remark 1 and Lemma 2.2, $\hat{w}_2(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{\delta_2}$.

On the other hand,

$$\begin{aligned}&\hat{w}_2(z_1,z_2)^T\left[-B_{11}^{(2)}C^{(1)}z_1 - B_{21}^{(2)}C^{(1)}z_2 \quad \left(B_{12}^{(2)} + B_{11}^{(2)}D^{(1)}\right)z_1 + \left(B_{22}^{(2)} + B_{21}^{(2)}D^{(1)}\right)z_2\right] + \\ &+ \hat{w}_1(z_1,z_2)^T\left[I_{\delta_1} - A_1^{(1)}z_1 - A_2^{(1)}z_2 \quad B_1^{(1)}z_1 + B_2^{(1)}z_2\right] \in \mathbb{F}[z_1,z_2]^{1\times(\delta_1+k)}\end{aligned}$$

which implies that

$$\hat{w}_1(z_1,z_2)^T\left[I_{\delta_1} - A_1^{(1)}z_1 - A_2^{(1)}z_2 \quad B_1^{(1)}z_1 + B_2^{(1)}z_2\right] \in \mathbb{F}[z_1,z_2]^{1\times(\delta_1+k)}$$

and, since Σ_1 is modally reachable, $\hat{w}_1(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{\delta_1}$. Therefore $\hat{w}(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{1\times(\delta_1+\delta_2)}$ and, by Lemma 2.2, $R(z_1,z_2)$ is ℓFP and thus Σ is modally reachable.

Using a similar reasoning the proof of (2) readily follows. □

The next result follows from previous theorem and Proposition 2.14 and provides conditions for obtaining a systematic convolutional code.

Corollary 3.6: *For $i = 1,2$, let $\mathcal{C}_i$ be a 2D convolutional code with ISO representation $\Sigma_i = \left(A_1^{(i)}, A_2^{(i)}, B_1^{(i)}, B_2^{(i)}, C^{(i)}, D^{(i)}\right)$ of dimension δ_i. Suppose that the matrix $I_{\delta_2} - A_1^{(2)}z_1 - A_2^{(2)}z_2$ is unimodular and $\mathcal{C}_1$ is systematic. Then the series concatenation of $\mathcal{C}_1$ and $\mathcal{C}_2$ is systematic.*

However, the next example shows that the concatenation of two systematic 2D convolutional codes does not necessarily yield a systematic 2D convolutional code.

Example 3.7: Let α be a primitive element of the Galois field $\mathbb{F} = GF(8)$ with $\alpha^3+\alpha+1 = 0$, and consider, for $i = 1,2$, the 2D convolutional code $\mathcal{C}_i$ with ISO representation $\Sigma_i = \left(A_1^{(i)}, A_2^{(i)}, B_1^{(i)}, B_2^{(i)}, C^{(i)}, D^{(i)}\right)$, where

$$A_1^{(1)} = A_2^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix}, \qquad B_1^{(1)} = B_2^{(1)} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix},$$

$$C^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad D^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$A_1^{(2)} = A_2^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_1^{(2)} = B_2^{(2)} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \alpha \end{bmatrix},$$

$$C^{(2)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad D^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Note that $\mathcal{C}_1$ and $\mathcal{C}_2$ are systematic. In fact, the corresponding matrices defined by (3)

$$X_1(z_1,z_2) = \begin{bmatrix} 1 & z_1+z_2 & 0 & 0 & 0 \\ 0 & 1+\alpha(z_1+z_2) & \alpha & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix},$$

$$X_2(z_1,z_2) = \begin{bmatrix} 1+z_1+z_2 & 0 & z_1+z_2 & z_1+z_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & z_1+z_2 & \alpha(z_1+z_2) & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix},$$

both have a unimodular submatrix of order 4 (columns $1,2,3$ and 5 for matrix $X_1(z_1,z_2)$ and for matrix $X_2(z_1,z_2)$).

Let $\mathcal{C}$ be the series concatenation of $\mathcal{C}_1$ and $\mathcal{C}_2$; then, by Theorem 3.1, the corresponding matrix $X(z_1, z_2)$ defined by (3) is

$$\begin{bmatrix} 1+z_1+z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & z_1+z_2 & z_1+z_2 & (1+\alpha)(z_1+z_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & z_1+z_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+z_1+z_2 & \alpha(z_1+z_2) & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} = N(z_1, z_2)\overline{X}(z_1, z_2)$$

with

$$N(z_1, z_2) = \begin{bmatrix} 1+z_1+z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which is not unimodular and therefore $X(z_1, z_2)$ is is not ℓFP.

To conclude the paper we present a lower bound on the distance of the concatenated code in terms of the distance of $\mathcal{C}_1$ and the distance of the set constituted by the parity vectors corresponding to the codewords of $\mathcal{C}_2$.

For $i = 1, 2$, let $\Sigma_i = \left(A_1^{(i)}, A_2^{(i)}, B_1^{(i)}, B_2^{(i)}, C^{(i)}, D^{(i)}\right)$ be an ISO representation of the 2D systematic code $\mathcal{C}_i$, with dimension δ_i, where $I_{\delta_2} - A_1^{(2)} z_1 - A_2^{(2)} z_2$ is unimodular. Let also $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ be the 2D linear system defined in Theorem 3.1 and $\mathcal{C} = \mathcal{C}\left(A_1, A_2, B_1, B_2, C, D\right)$ be the corresponding code. Then, a codeword of $\mathcal{C}$ is of the form

$$\hat{v}(z_1, z_2) = \begin{bmatrix} \hat{y}_2(z_1, z_2) \\ \hat{v}_1(z_1, z_2) \end{bmatrix}$$

where $\hat{v}_1(z_1, z_2) \in \mathcal{C}_1$ and, by Proposition 2.14, $\hat{y}_2(z_1, z_2) = T_2(z_1, z_2)\hat{v}_1(z_1, z_2)$ with

$$T_2(z_1, z_2) = C^{(2)}\left(I_{\delta_2} - A_1^{(2)} z_1 - A_2^{(2)} z_2\right)^{-1}\left(B_1^{(2)} z_1 + B_2^{(2)} z_2\right) + D^{(2)} \in \mathbb{F}[z_1, z_2]^{(n-m)\times m}.$$

Moreover, let us assume that

$$\ker T_2(z_1, z_2) \cap \mathcal{C}_1 = \{0\}.$$

Then we obtain that

$$\mathrm{dist}(\mathcal{C}) \geq \mathrm{dist}(\mathcal{C}_1) + \mathrm{dist}(\mathrm{Im}\, T_2(z_1, z_2))$$

where $\mathrm{dist}(\mathrm{Im}\ T_2(z_1, z_2)) = \min\{\mathrm{wt}(\hat{y}) \mid \hat{y}(z_1, z_2) \in \mathrm{Im}\, T_2(z_1, z_2),\ \text{with}\ \hat{y}(z_1, z_2) \neq 0\}.$

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